Equations of planes in space

You should be familiar with equations of lines in the plane. From this experience, you know that the equation of a line in the plane is a *linear equation in two variables*. We'll use x and y as the two variables. As an example, consider the equation

$$3x + 4y - 8 = 0.$$

This form of the equation is called the *standard form*. We can algebraically manipulate this into other forms such as the *slope-intercept form*

$$y = -\frac{3}{4}x + 2$$

or a *point-slope form*

$$y - 5 = -\frac{3}{4}(x + 4).$$

[Note that there are many point-slope forms depending on which point we choose to focus attention. Here, the point $(-4, 5)$ was chosen as the focus of attention.] Each of these forms is useful in different contexts. In calculus, a point-slope form is often useful in writing the equation of a tangent line since we most often have information about a point on the tangent (from the function) and the slope of the tangent line (from the derivative of the function).

More generally, we can express these forms as

$Ax + By + C = 0$	standard form
$y = mx + b$	slope-intercept form
$y - y_0 = m(x - x_0)$	point-slope form

You are probably comfortable with reading off geometric information from the latter two equations. We will see later that the constants A and B in the standard form can also be given direct geometric interpretation.

Planes in space are described by *linear equations in three variables*. For example, consider the equation

$$3x + 4y - 2z - 12 = 0.$$

The set of all points with cartesian coordinates (x, y, z) that satisfy this equation form a particular plane. We can read off geometric information about this plane if we solve for z to get

$$z = \frac{3}{2}x + 2y - 6.$$

This is the *slopes-intercept* form for the equation of this plane. Note that *slopes* is plural here since we have *two* slopes. The coefficient $3/2$ is the x-slope and the coefficient 2 is the y-slope. We'll denote these m_x and m_y so here we have

$$m_x = \frac{3}{2} \qquad \text{and} \qquad m_y = 2.$$

The x-slope is a "rise over run" with y held constant and, in similar fashion, the y-slope is "rise over run" with x held constant. To be more detailed, we have

$$m_x = \frac{\text{rise in } z}{\text{run in } x} \quad \text{with } y \text{ held constant}$$

and

$$m_y = \frac{\text{rise in } z}{\text{run in } y} \quad \text{with } x \text{ held constant.}$$

[Note that the rise is a change in z for both of these since we have singled out the z coordinate by solving the original equation for this variable.] So, for this example, we have a rise of 3 units in the z direction for any run of 2 units in the x direction with y kept constant. Similarly, by thinking of 2 as $2/1$, we have a rise of 2 units in the z direction for any run of 1 unit in the y direction with x kept constant.

The two slopes $m_x = 3/2$ and $m_y = 2$ give us the orientation of the plane. The constant term -6 in the equation is the z-intercept (since the equation gives $z = -6$ with $x = 0$ and $y = 0$). The z-intercept picks out one particular plane in the stack of parallel planes having slopes $m_x = 3/2$ and $m_y = 2$.

More generally, we can express the equation of a plane in any one of several forms:

$$\begin{array}{rl} Ax + By + Cz + D = 0 & \text{standard form} \\ z = m_x x + m_y y + b & \text{slopes-intercept form} \\ z - z_0 = m_x(x - x_0) + m_y(y - y_0) & \text{point-slopes form} \end{array}$$

Exercises

1. Determine which, if any, of the following points are on the plane having equation $2x - y + 6z = 14$.

   (a) $(5, -4, 0)$ (b) $(1, 6, 2)$ (c) $(2, 8, 3)$

2. Determine the x-intercept, the y-intercept, and the z-intercept of the plane having equation $2x - y + 6z = 14$.
3. Determine the slopes of the plane having equation $2x - y + 6z = 14$.
4. Find the standard form equation for the plane containing the point $(2, -6, 1)$ with slopes $m_x = 3$ and $m_y = -2$.
5. Find an equation for the plane that contains the points $(0, 0, 0)$, $(2, 0, 6)$, and $(0, 5, 20)$.
6. Find an equation for the plane that contains the points $(0, 0, 0)$, $(0, 4, -8)$, and $(3, 0, 6)$.
7. Find an equation for the plane that contains the points $(1, 3, 2)$, $(1, 7, 10)$, and $(3, 3, 8)$.
8. Find an equation for the plane that contains the points $(7, 2, 1)$, $(5, 2, -4)$, and $(5, -2, 10)$.
9. (*Challenge problem*) Find an equation for the plane that contains the points $(1, 3, 2)$, $(1, 7, 10)$, and $(4, 2, 1)$.